ASSUMING ONE PARAMETER GROUP OF CONFORMAL MOTIONS : EXACT PLANE SYMMETRIC PERFECT FLUID

Dr. Amit kumar Srivastava,
Department of Physics,
D.A-V. College,
Kanpur (U.P.), India

ABSTRACT

Our distribution is nonstatic, shearing and expanding. Some solutions are homogeneous. If one leaves the assumption of one parameter group of conformal motions, then perfect fluid with heat flux distribution is obtained.

Perfect fluid with heat flux distributions, in plane symmetry admitting a one parameter group of conformal motions, are not admitted. We have proved that perfect fluid with heat flux distributions, in plane symmetry admitting a one parameter group pf conformal motions, are not admitted. Hence, we have investigated exact plane symmetric perfect fluid solutions of Einstein equations assuming one parameter group of conformal motions. We have discussed the geometrical and physical properties of some particular solutions so obtained. Hence, we have investigated exact plane symmetric perfect fluid solutions of Einstein equations assuming one parameter group of conformal motions. Here the geometrical and physical properties of some particular solutions so obtained.

Keywords: *Nonstatic perfect fluid, Heat flux distribution, Non thermalised perfect fluid*

INTRODUCTION

A space time that possesses the three parameters group of motions of the Euclidean plane is said to have plane symmetry and is known as a plane symmetric space time. Such space times have many properties similar to those of spherically symmetric and hyperbolic symmetric ones. For examples, they obey the Birkoff theorem as given by (*Taub 1951*), vacuum solutions of the field equations for such space times are of the same Petrov class as the corresponding spherically symmetric ones as investigated by (*Ehlers and Kundt 1962*). Plane symmetric space times with source terms as perfect fluid, have been investigated due to possible applications to Astrophysics and Cosmology as discussed by (*Taub 1972*). We assume that the space time with perfect fluid and heat flux as the source of gravitational field, admits, besides the plane symmetry, a non-parameter group of conformal motions i.e.

1. $$L_\xi \, g_{\alpha\beta} = \xi_{\alpha;\beta} + \xi_{\beta;\alpha} = \Psi \, g_{\alpha\beta}$$

where L denotes the Lie derivative and Ψ is an arbitrary function of the coordinates. Under these assumptions solutions with spherical symmetry, have been obtained by (*Herrera and Leon 1985*). Under the above assumptions, we are able to integrate the Einstein equations and physical features of the solutions are investigated. It is obtained that the real null congruences $\underline{l}$ and $\underline{n}$ are geodesic and shear free, the solution are algebraically Petrov type D and belongs to class I of (*Winwright 1977*) classification scheme.

THE METRIC AND FIELD EQUATIONS

Let us consider a nonstatic perfect fluid with heat flux distribution having plane symmetry with the line element.

2. $$ds^2 = e^{2v} dt^2 - e^{2\lambda} dx^2 - e^{2\mu} (dy^2 + dz^2)$$

with

3. $$x^0 = t, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z$$

and v, λ, μ are functions of x and t i.e.

4. $$v = v \,(x, t), \quad \lambda = \lambda \,(x, t), \quad \mu = \mu \,(x, t)$$

The matter content of the spacetime is assumed to be a non thermalised perfect fluid described by the energy momentum tensor

5. $$T^{\alpha}_{\ \beta} = (\rho + p) u^\alpha u_\beta - p \, \delta^\alpha_{\ \beta} + (q^\alpha u_\beta + q_\beta u^\alpha)$$

6. $$u_\alpha u^\alpha = 1$$

7. $$u_\alpha q^\alpha = 0,$$

where ρ and p denote the energy density and fluid pressure of matter distribution. u^α denotes unit time like flow vector of the fluid and q^α denotes the spacelike heat flow vector orthogonal to u^α. We assume the coordinates to be commoving and we have

8. $$u^\alpha = \delta^\alpha_{\ 0} \, e^{-v}$$

Now we assume that the space time admits one parameter group of conformal motion given by eq. (1) with the restriction that the vector q is orthogonal to fluid velocity vector given by eq. (7). The equation (7) in the view of plane symmetry implies that

$q^0 = 0$

$q^2 = 0$

9. $q^3 = 0$

$q^1 = q$ (say).

The Einstein field equations are

10. $G^{\alpha}_{\beta} = 8\pi \; T^{\alpha}_{\beta}$

where we have taken the units such that

11. $c = G = 1$

The equation (10) reads

12. $$-2e^{-2\lambda}(\mu'' + \frac{3}{2}\mu^2 - \lambda'\mu') + e^{-2\nu}(\dot{\mu}^2 + 2\dot{\mu}\dot{\lambda})$$

$= 8 T^o_o$

$= 8\pi\rho$

13. $$-e^{2\lambda}(\mu'^2 + 2\mu'\nu') + 2e^{-2\nu}(\ddot{\mu} - \dot{\mu}\dot{\nu} + \frac{3}{2}\dot{\mu}^2) = 8\pi T^1_1 = -8\pi p$$

14. $$-e^{-2\lambda}(\nu'' + \nu'^2 - \lambda'\nu' + \mu'' + \mu'^2 + \mu'\nu' - \lambda'\mu')$$

$$+e^{-2\nu}(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}'\dot{\nu} + \ddot{\mu} + \dot{\mu}^2 + \dot{\mu}\dot{\lambda} - \dot{\mu}\dot{\nu})$$

$$= 8\pi T^2_2 = 8\pi p,$$

$$= 8\pi T^3_3,$$

15. $$2e^{-(\nu+\lambda)}(\dot{\mu}\mu' - \dot{\lambda}\mu' + \dot{\mu}' - \dot{\mu}\nu')$$

$$= 8\pi T^o_1$$

$$= 8\pi q \; e^{-\nu}$$

where dots and primes denote differentiation with respect to t and x respectively.

Subsequently it is desired that

16. $T^o_1 \neq 0$

so that space time given by eq. (2) may sustain presence of heat flow.

In view of eqs. (1) and (9) one obtains

17. $q\nu' = \Psi/2$

18. $\dot{q} = 0$

19. $q' + q\lambda' = \Psi/2$

20. $q\mu' = \Psi/2$

From Eqs. (17) and (20), we get

21. $\nu - \mu = f_1(t)$,

$f_1(t)$ is an arbitrary function of t.

one obtains, by differentiating eqs. (19) and (20).

22. $$\dot{\lambda}' = \dot{\mu}'$$

It follows from eq. (22) that

23. $\lambda = \mu + f_2(t) + g_1(x)$

where $f_2(t)$ and $g_1(x)$ are arbitrary functions of their arguments.

Let us execute a coordinate transformation of the form

24. $$t = t(\bar{t})$$

25. $$r = r(\bar{r})$$

and one may select.

$$f_1(t) = g_1(x) = 0$$

without any loss of generality.

Hence, we get

26. $\mu = \nu$,

27. $\lambda = \mu + f(t)$

In view of eqs. (26) - (27) from eqs. (19) and (20), one easily obtain

28. $q' = 0$

which is view of eq. (18) gives that

29. $q = A =$ a constant

or

30. $\Psi = 2A\,v'$

Let us now consider the field equation (15) in view of Eqs. (26) - (27).

31. $$2e^{-\lambda}(-\dot{\lambda}\,\mu' + \dot{\mu}') = 8\pi\, q = q\pi\, A$$

or

32. $$2e^{-\lambda}(-\dot{\lambda}\,\lambda' + \dot{\lambda}') = 8\pi\, A$$

Let us define a new variable.

33. $$Z \equiv e^{-\lambda}$$

In view of Eq. (33) the Eq. (32) reduces

34. $$\dot{Z}' = -4\pi\, A,$$

whose solution assumes the form

35. $$Z = e^{-\lambda} = -4Axt + h(x) + g(t),$$

where h and g are arbitrary functions of their arguments. Hence, we obtain

36. $e^{-\nu} = e^{-\mu}$
$= e^{f} [- 4 Axt + h (x) + g (t)]$

It is an important to note that the presence of heat flux is governed by the constant A. Hence, A ≠ 0. If g = 0 we recovery the result of perfect fluid distribution. The line element (2) maybe expressed in terms of the functions, and we obtain.

37. $$ds^2 = \frac{e^{-2f}}{[-4\pi Axt + h + g]^2}[dt^2 - e^{2f}dx^2 - dy^2 - dz^2]$$

INTEGRATION OF THE FIELD EQUATIONS

We have investigated that $q^1 = q = A$ (a constant). Hence, plane symmetric space time, admitting as one parameter – group of conformal motions, does not allow perfect fluid distributions with heat flux. Under these conditions

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38. $$e^{-\nu} = e^{-\mu} = e^{f}(h + g)$$

Hence, the line element and field equations may be recasted in terms of functions f, h and g and we obtain –

39. $$ds^2 = \frac{e^{-2f}}{[h + g]^2}[dt^2 - e^{2f}dx^2 - dy^2 - dz^2]$$

40. $$8\pi\rho = 3(\dot{g}^2 e^{2f} - h'^2) + 2e^{-\lambda}(h'' + 2\dot{f}\dot{g}e^{2f}) + e^{2(f-\lambda)}\dot{f}^2$$

41. $$-8\pi p = 3(\dot{g}^2 e^{2f} - h'^2) + 2e^{f-\lambda}(\dot{g}\dot{f} - \ddot{g}) + e^{2(f-\lambda)}(\dot{f}^2 - 2\ddot{f}),$$

42. $$-8\pi p = 3\left(\dot{g}^2 e^{2f} - h'^2\right) + 2\bar{e}^{\lambda}(h'' - \ddot{g}e^{2f}) - e^{2(f-\lambda)}\ddot{f}$$

If view of eqs. (41) and (42), one obtains

43. $$\frac{e^{2f}}{2}(\ddot{f} - \dot{f}^2) = e^{\lambda}(e^{2f}\dot{g}\dot{f} - h'')$$

Let us put

44. $$\Phi(t) = \frac{e^{2f}}{2}(\ddot{f} - \dot{f}^2)$$

Hence, we obtain

45. $$e^{\lambda}(e^{2f}\dot{g}\dot{f} - h'') = \Phi(t)$$

Let us differentiate with x, we obtain

46. $e^{\lambda}\lambda^{'}(e^{2f}\dot{g}\dot{f}-h^{''})+e^{\lambda}(-h^{'''})=0$

or

47. $\lambda^{'}\Phi(t) = h^{'}e^{\lambda}$

Now, in absence of q, we get

48. $Z = \bar{e}^{\lambda} = h(x) + g(t)$

Differentiating with rest to x, one obtains

49. $e^{-\lambda}(-\lambda^{'}) = h^{'}(x) = h^{'}$

or

50. $\lambda' = h' e^{\lambda}$

In view of Eqs. (47) and (50), we get

51. $\Phi(t) = -\frac{h^{'''}}{h^{'}}$

Eq. (51) implies that

52. $\Phi(t)$ = Constant = B.

The first integration of Eq. (51) reads

53. $h'' + Bh = C$

Where C is constant.

In view of Eqs. (44), (48), (51), the Eq. (45) reduces to

54. $e^{2f}\dot{g}\dot{f} = Bg + C$

The first integral of Eq. (44) reads

55. $\dot{f}^{2} = De^{2f} - Be^{-2f}$

Where D is another constant.

Let us integrate Eq. (53) to obtain

56. $h'^{2} = 2Ch - Bh^{2} + E$

Where E is constant.

Let us define a new variable

57. $Q(t, x) = e^{f(t)} / h(x) + g(t)$

or

58. $h = (\frac{e^{f}}{Q} - g)$,

using Eq. (58), in Eq. (56), we get

59. $h^{'2} = 2c(\frac{e^{f}}{Q} - g) - B(\frac{e^{f}}{Q} - g)^{2} + E$

$$= \frac{Q'^2}{Q^2}(h+g)$$

In view of these equations, one way easily obtain the expressions for pressure and density as

60. $$8\pi\, p = \frac{De^{2f}}{Q^2} - 2\frac{De^f}{Q\dot{f}^2}(C+Bg) - F(t)$$

61. $$8\pi\, p = \frac{De^{2f}}{Q^2} + F(t)$$

where

62. $$F(t) = 3\left[2\,Cg + Bg^2 - E + \left(\frac{C+Bg}{fe^f}\right)^2\right]$$

For the case $\rho \geq P$, the following inequality must be satisfied.

63. $$2De^f\frac{(C+Bg)}{Q\dot{f}^2} + 2F \geq 0$$

In order to have pressure positive, we have keep $D \neq 0$ and $f \neq 0$.

KINEMATICAL PARAMETERS

The kinematical parameters of the fluid are obtained as

64. $$\theta = -e^f[3\dot{g} + 2\dot{f}(h+g)],$$

65. $$\sigma = \frac{\dot{f}}{\sqrt{3}}e^{f-\lambda} = \frac{\dot{f}}{\sqrt{3}}e^f(h+g),$$

66. $$a^\alpha = \dot{U}^\alpha = -\delta_1^\alpha h'(h+g)$$

The gravitational field, admitting a one parameter group of conformal motions, is of Petrov type D. The nonvanishing Weyl coefficient is Ψ_2 as

67. $$\psi_2 = \frac{1}{3}e^{-2\mu}(\ddot{f} + \dot{f}^2)$$

SOME PROPERTIES OF SOLUTIONS

By different choices of constants B, C, D and E one may obtain different particular solutions. Now we will discuss some particular cases :

Case 1 For $B = C = 0$

In these choices e.g. (56), (62) and (63) imply that

68. $h'^{2} = E$

69. $F(t) = -3E$

70. $F(t) = 0$

which means

71. $F(t) = E = 0$

72. $h(x) = M$ = constant (integration), and in view of eq. (54)

73. $g(t) = N$ = Integration constant

In above choice eq. (55) reduces to

74. $$D = \frac{\dot{f}^2}{e^{2f}}$$

Let us assume

75. $D = w^2$

and one obtain

76. $$\dot{f} = \mathrm{w}e^{f}$$

Let us redefine the origin of time.

77. $$e^{f} = \frac{1}{\mathrm{w}t}$$

78. $$p = \rho = \frac{[\mathrm{w}(M+N)]^2}{8\pi t^4}$$

$$= \frac{L}{t^4}$$

where

79. $$L = \frac{[\mathrm{w}(M+N)]^2}{8\pi}$$

Hence, the metric assumes the form

80. $$ds^2 = (Ht)^2\left[dt^2 - \frac{dx^2}{(\mathrm{w}t)^2} - (dy^2 + dz^2)\right]$$

where

81. $$H = \left(\frac{w}{M + N}\right)$$

The Weyl coefficient Ψ_2 reads

$$\psi_2 = \frac{2}{3}\left(\frac{1}{H^2 t^2}\right)$$

Hence, the generator ξ is space like killing vector. The kinematical parameters read

82. $$\sigma = (\sqrt{3}\, Ht^2)^{-1}$$

83. $$\theta = 2(Ht^2)^{-1},$$

84. $a^{\alpha} = 0$

The solution represents stiff matter and expanding configuration. Shear and expansion both decreases and vanish asymptotically with time.

Case 2 For B = D = 0

In the above choice, one obtains.

85. C = 0

and

86. $h = k\ x + a, \quad k^2 = E$

where a is constant of integration.

Hence, the metric reads

87. $ds^2 = (kx + a + g)^{-2} [dt^2 - dx^2 - dy^2 - dz^2]$

The expressions for density and pressure are

88. $g \pi \rho = 3(\dot{g}^2 - k^2)$

89. $$g\pi\, p = -3(\dot{g}^2 - k^2) + 2\ddot{g}(kx + a + g)^{-1}$$

such that

90. $$(\rho + p) = \frac{1}{4}\ddot{g}(kx + a + g)^{-1}$$

The kinematical parameters are

$$\theta = -3\dot{g}$$

91. $\sigma = 0$

$a^y = a^z = a^t = 0$

$a^x = -k\ (kx + a + g)$

For K = 0, acceleration vanishes and pressure distribution becomes homogeneous.

Case 3 For C = g = 0

In above condition Eq. (53) gives

92. $$h = \begin{cases} A_1 \sin bx + A_2 \cos bx, & B = b^2 \\ A_1 \sinh bx + A_2 \cosh bx, & B = -b^2 \end{cases}$$

where A_1, A_2, B_1, B_2 are integration constants.

In this case pressure and density read

93. $$8\pi\, p = \frac{De^{2f}}{Q^2} + 3E$$

94. $$8\pi\, \rho = \frac{De^{2f}}{Q^2} - 3E$$

where E < 0, D > 0.

Again one obtains

95. $$8\pi\,(\rho + p) = \frac{De^{2f}}{Q^2}$$

Let us define a new function Q_o as

96. $$Q_o = \sqrt{-\frac{D}{3E}}\, e^f$$

In view of eq. (55), Q_o satisfies

97. $$\ddot{Q}_o = -6E\, Q_o^3$$

A particular solution of eq. (97) gives

98. $$Q_o = -\frac{1}{\sqrt{-3Et}},$$

which, from eq. (96) gives

99. $$e^f = -\frac{1}{\sqrt{Dt}}$$

Hence, one obtains explicit expressions for density and pressure

100. $$8\pi\rho \begin{cases} \dfrac{A_1 \sin \mathrm{bx} + \mathrm{A}_2 \cos bx,}{Dt^4} - 3E, & B = b^2 \\ \dfrac{B_1 \sinh \mathrm{bx} + \mathrm{B}_2 \cosh bx,}{Dt^4} - 3E, & B = -b^2 \end{cases}$$

101. $$8\pi\, p = 8\pi\rho + GE$$

The kinematical parameters read

102. $$\theta = \frac{-2h}{\sqrt{D}\, t^2}$$

103. $$\sigma = \frac{h}{\sqrt{3D}\, t^2}$$

104. $$a^t = a^y = a^z = 0$$
$$a^x = -h\, h'$$

The Weyl coefficient is

105. $$\psi_2 = \frac{2}{3D\, t^4} \begin{cases} (A_1 \sin \mathrm{bx} + \mathrm{A}_2 \cos bx)^2, & B = b^2 \\ (B_1 \sinh \mathrm{bx} + \mathrm{B}_2 \cosh bx)^2, & B = -b^2 \end{cases}$$

It is obvious that both density and pressure distributions are inhomogeneous. Shear and expansion of the fluid decrease to zero asymptotically with time.

CONCLUDING REMARKS

We have proved that perfect fluid with heat flux distributions, in plane symmetry admitting a one parameter group pf conformal motions, are not admitted. Hence, we have investigated exact plane symmetric perfect fluid solutions of Einstein equations assuming one parameter group of conformal motions. We have discussed the geometrical and physical properties of some particular solutions so obtained. Our distribution is nonstatic, shearing and expanding. Some solutions are homogeneous. If one leaves the assumption of one parameter group of conformal motions, then perfect fluid with heat flux distribution is obtained.

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