EINSTEIN-MAXWELL EQUATIONS IN PERFECT MAGNETOFLUID AS THE SOURCE OF ENERGY MOMENTUM TENSOR

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ABSTRACT

The condition under which the simple fluid may be interpreted as a two fluid mixture are discussed. The influence of the field near the singularity and at later stages of the expansion is presented. A subclass of these models approaches homogeneity and isotropy for large cosmological times. We have presented an algorithm for obtaining exact solutions of perfect magnetofluid in Ruban's background. We have obtained a new class of expanding inhomogeneous solutions generalising the dust model found by Ruban.A method for obtaining exact solutions for the Einstein-Maxwell equations in Ruban's background is presented, with perfect magnetofluid as the source of energy momentum tensor. In the most general case the cosmological constant is nonzero and the matter content is perfect magnetofluid only with magnetic field. By considering the case of simple fluid, a class of expanding inhomogeneous solutions generalising the dust model obtained by Ruban and the Doroshkevich magnetic universe is obtained.

INTRODUCTION

Since the discovery of the Szekeres (1975) dust filled universe increasing attention has been paid to inhomogeneous cosmological models such as Szafron (1977), Pollack and Cadderni (1980), Goode and Wainwright (1982), Lima (1986) and Raj and Singh (1987). We have presented an algorithm for obtaining exact solutions of the Einstein-Maxwell equations in Ruban's background.

In astrophysical and cosmological problems, the assumption of a primeval magnetic field has many interesting consequences. In fact, such a field could play an crucial role on the structure of formation process, in the origin of the galactic and intergalactic magnetic field, as well as to alter significantly the underlying geometric structure of the universe, at least in the early stages of the cosmic evolution as presented by Hoyle (1958), Doroshkevich (1965), Zeldovich (1965, 1970), Fujimoto et al (1971) and Reinhardt et al (1970). Dynamical effects produced by magnetic field were discussed by several authors, firstly in the framework of homogeneous axially symmetric models by Stewart and Ellis (1972), Vajk and Eltgroth (1970), Ruban (1982), Spokoiny (1982), Szekeres (1975), Bonnor et al (1976).

THE METRIC AND EINSTEIN FIELD EQUATIONS WITH COSMOLOGICAL CONSTANT Λ

Let us consider Ruban (1969) line element

$$= dt^2 - Q^2\left(x,t\right)dx^2 - R^2\left(t\right)\left(dy^2 + H^2 dz^2\right) \tag{1}$$

with

$$H\left(y\right) = \frac{\sin\sqrt{k}y}{\sqrt{k}} = \begin{cases} \sin y & \text{if } k = 1 \\ y & \text{if } k = 0 \\ \sinh y & \text{if } k = -1 \end{cases} \tag{2}$$

where k be the curvature parameter of homogeneous 2-spaces t and x constant. The functions Q and R are free and will be obtained by the Einstein field equation with cosmological constant Λ in units $8\pi G = C = 1.$

$$R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R + \Lambda g^{\alpha\beta} = T_m^{\alpha\beta} + T_f^{\alpha\beta}, \tag{3}$$

where $T_m^{\alpha\beta}$ be the energy momentum tensor of material medium and $T_f^{\alpha\beta}$ as the energy momentum tensor of electromagnetic field. Hence, for perfect magnetofluid

$$T_{\alpha\beta}^{f} = \left[\left(p+\rho\right)u_\alpha u_\beta - pg_{\alpha\beta}\right], \tag{4}$$

and

$$\tau_{\alpha\beta}^{f} = \mu\left[\left(u_\alpha u_\beta - \tfrac{1}{2}g_{\alpha\beta}\right)\left|\underset{\sim}{h}\right|^2 - h_\alpha h_\beta\right], \tag{5}$$

where the vector h_ρ being a spacelike vector such that

$$\left|\underset{\sim}{h}\right|^2 = -h_\rho h^\rho \geq 0, \tag{6}$$

with μ as the magnetic permeability as a given constant.

In the case of perfect magnetohydrodynamics, the electric current J is not known and the Maxwell equations are

$$\left(u^\alpha h^\alpha - u^\beta h^\alpha\right)_{;\alpha} = 0 \tag{7}$$

We expand these equations

$$h^\beta u^\alpha_{;\alpha} + u^\alpha h^\beta_{;\alpha} - h^\alpha u^\beta_{;\alpha} - u^\beta h^\alpha_{;\alpha} = 0, \tag{8}$$

with $h^\beta u_\beta = 0$ and $u_\beta u^\beta = 1.$

According to equations (4) and (5), it is possible to write the energy momentum tensor

$$T_{\alpha\beta} = \left[\left(p + \rho + \mu \left| \underset{\sim}{h} \right|^2 \right) u_\alpha u_\beta - \left(p + \frac{1}{2} \mu \left| \underset{\sim}{h} \right|^2 \right) g_{\alpha\beta} - \mu h_\alpha h_\beta , \quad (9)$$

where

$$P = p + \frac{1}{2} \mu \left| \underset{\sim}{h} \right|^2 , \quad (10)$$

$$W = \rho + \frac{1}{2} \mu \left| \underset{\sim}{h} \right|^2 . \quad (11)$$

Hence,

$$T_{\alpha\beta} = \left(P + W \right) u_\alpha u_\beta - P g_{\alpha\beta} - \mu h_\alpha h_\beta . \quad (12)$$

Perfect magnetohydrodynamics is the study of the properties of a perfect fluid with an infinite conductivity $\sigma = \infty$. The electric current $\underset{\sim}{J}$, and thus the product $\sigma \underset{\sim}{e}$ being essentially finite, we have in this case $\underset{\sim}{e} = 0$. The electromagnetic field is reduced to a magnetic field $\underset{\sim}{h}$ with respect to the velocity of the considered fluid.

The Einstein field equations with cosmological constant Λ i.e.

$$R_{\alpha\beta} - \left(\frac{1}{2} R - \Lambda \right) g_{\alpha\beta} = T_{\alpha\beta} , \quad (13)$$

reduces to

$$QR^2 T_{00} = Q\dot{R}^2 + 2R\dot{Q}\dot{R} + kQ - \Lambda QR^2 , \quad (14)$$

$$Q^{-2} R^2 T_{11} = -\ddot{R}\partial R - \dot{R}^2 - k + \Lambda R^2 , \quad (15)$$

$$QR^{-1} T_{22} = -Q\ddot{R} - \dot{Q}\dot{R} - \ddot{Q}R + \Lambda QR , \quad (16)$$

$$QR^{-1} H^{-2} T_{33} = -Q\ddot{R} - \dot{Q}\dot{R} - \ddot{Q}R + \Lambda QR , \quad (17)$$

where an overdot means time derivative.

Let us consider comoving frame

$$u = \delta_0^\mu . \quad (18)$$

In view of eq. (12), one obtains

$$\left.\begin{array}{l} T_{00} = W, T_{11} = Q^2\left(P - \mu\left|\underset{\sim}{h}\right|^2\right) \\ T_{22} = R^2 P, T_{33} = PR^2 H^2 \end{array}\right\} \tag{19}$$

Now for $h^{\beta} u_{\beta} = 0,$ we obtain

$$h^0 = 0$$

Let us assume

$$h^1 \neq 0, \quad h^2 = h^3 = h^0 = 0. \tag{20}$$

Eq. (7) is identically satisfied, and we get

$$\left(h^1\right)^2 = \left|\underset{\sim}{h}\right|^2 \bar{Q}^2 \tag{21}$$

and

$$\left(h^1\right)^2 = -Q^2 \left|\underset{\sim}{h}\right|^2 \tag{22}$$

Hence, Einstein field equations assumes the form

$$W = \frac{2\dot{R}\dot{Q}}{RQ} + \frac{\dot{R}^2 + k}{R^2} - \Lambda, \tag{23}$$

$$P - \mu\left|\underset{\sim}{h}\right|^2 = \frac{-2\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} - \frac{k}{R^2} + \Lambda, \tag{24}$$

$$\mu\left|\underset{\sim}{h}\right|^2 QR = \left[\ddot{R} + \frac{\dot{R}^2 + k}{R}\right] Q - \ddot{Q}R - \dot{Q}\dot{R}. \tag{25}$$

The system of eqs. (23) - (25) is indeterminate because there are three differential equations and four unknown quantities, namely W, P, R and Q in our description. Moreover, the net pressure P is a function of t alone whereas the net energy density W is a function of t and x. Therefore, the usual equation of state maynot be used without loss of generality. Hence, this is the same problem appearing for the first time in Szekeres' type models. Szafron (1977) suggested an algorithm for obtaining exact solutions, as follows:

(i) Specify the net pressure $P = P\left(R, \dot{R}, \ddot{R}, \Lambda\right)$ and solve it for $R = R(t),$

(ii) Obtain $Q = Q(R, x)$ from eq. (25),

(iii) From eq. (23), evaluate the net energy density W by substituting the expressions of *Q(R,x)* and *R(t)*.

We remarks that if $P = \Lambda = 0,$ the solutions are simple inhomogeneous generalisations of Doroshkevich (1965) universe. If $\mu = 0$, Ruban's models are recovered and if $W(R) = C/R^2$ or $W(R) = 3(\gamma - 1)\left(\dot{R}^2 + k/R^2\right)$ where C and γ are constants then the models stand for two subclasses of Szekeres type solutions.

NEW CLASS OF SOLUTIONS

It is to be noted that if $\mu = 0$ or magnetic field vanishes, then from eq. (24) we recover FLRW models. Now, let us assume the choice of pressure

$$P = 3(\gamma - 1)\left(\dot{R}^2 + k\right)/R^2, \tag{26}$$

where the constant γ is identified as adiabatic index of the asymptotic in time with equation of state $P = (\gamma - 1)\rho.$

Let us put $\Lambda = 0,$ one obtains

$$R\ddot{R} + \frac{1}{2}(3\gamma - 2)\dot{R}^2 + \frac{1}{2}(3\gamma - 2)k - \frac{1}{2}\mu\left|\underset{\sim}{h}\right|^2 R^2 = 0 \tag{27}$$

The first integral of which reads

$$\dot{R}^2 = \left(\frac{R_0}{R}\right)^{3\gamma - 2} - k - \frac{1}{(4 - 3\gamma)}\mu\left(\left|\underset{\sim}{h}\right| R\right)^2 \text{ if } \gamma \neq 4/3 \tag{28}$$

and

$$\dot{R}^2 = \left(\frac{R_0}{R}\right)^2 - k - \mu\left(\frac{\left|\underset{\sim}{h}\right| R}{R_0}\right)^2 R_0^2 \ell n\left(\frac{R_0}{R}\right) \text{ if } \gamma = 4/3, \tag{29}$$

where R_0 be a γ -independent constant.

If $\mu = 0$ in eq. (28), then it is first integral for any γ and eq. (27) reduces to the FLRW differential equation. In this case, the solution of eq. (27) is valid for γ and *k* as given by Assad and Lima (1988), in terms of hypergeometric functions. However, if μ and *k* are both different from zero the method given there maynot be applied. Hence, for the sake of simplicity, we consider the quasi-Euclidean models *k = 0* and $\gamma \neq \frac{4}{3}$. In this case in view of Assad and Lima (1988), it is easy to show that the solution of eq. (27) or equivalently (28) is given by

$$t-t_0=\frac{2R_0}{4-3\gamma}\left(\frac{R}{R_0}\right)^{3\gamma/2}\left[1-\frac{\mu|\underset{\sim}{h}|^2R^2}{4-3\gamma}\left(\frac{R}{R_0}\right)^{3\gamma-4}\right]^{1/2}F_1$$

$$-\frac{2R_0}{4-3\gamma}\left[1-\frac{\mu|\underset{\sim}{h}|^2R^2}{4-3\gamma}\right]^{1/2}F_2. \qquad (30)$$

In the limit $\mu=0$ and in view of identity given by Abramowitz and Stegun (1965)

$$F(a;b;c;1)=\Gamma_{(c)}\Gamma_{(c-a-b)}/\Gamma_{(c-a)}\Gamma_{(c-b)}, \qquad (31)$$

gives

$$R=R_0\left[1+\frac{3}{2}\gamma(t-t_0)\right]^{2/3\gamma} \qquad (32)$$

which is the same FLRW models as given by Assad and Lima (1988), where $t_0=t(R)$, $F_2=F_1(R_0)$ and $F_1(h)$ is hypergeometric function

$$F_1=F\left[\frac{3\gamma-2}{3\gamma-4};1;\frac{3}{2};1-\frac{\mu|\underset{\sim}{h}|^2R^2}{4-3\gamma}\left(\frac{R}{R_0}\right)^{3\gamma-4}\right]. \qquad (33)$$

The solution of Q is given by Lima and Nobre (1990)

$$Q=\beta RF_3+\alpha R_0(R/R_0)^{(3\gamma-4)/2}F_4 \qquad (34)$$

where β and α are arbitrary functions of x and F_3, F_4 are two new hypergeometric functions

$$F_3=F\left[\frac{1-D}{2(3\gamma-4)},\frac{1+D}{2(3\gamma-4)},\frac{3\gamma-2}{2(3\gamma-4)},\frac{\mu|\underset{\sim}{h}|^2R^2}{4-3\gamma}\left(\frac{R}{R_0}\right)^{3\gamma-4}\right] \qquad (35)$$

$$F_A=F\left[\frac{3\gamma-5-D}{2(3\gamma-4)},\frac{3\gamma-5+D}{2(3\gamma-4)},\frac{9\gamma-14}{2(3\gamma-4)},\frac{\mu|\underset{\sim}{h}|^2R^2}{4-3\gamma}\left(\frac{R}{R_0}\right)^{3\gamma-4}\right] \qquad (36)$$

Now for μ = 0, the function Q reduces to

$$Q=\beta R+\alpha R_0(R/R_0)^{(3\gamma-4)/2} \qquad (37)$$

and we may evaluate

$$W = \frac{\dot{R}^2}{R^2}\left(1 + \frac{2RQ'}{Q}\right), \tag{38}$$

where $Q' = \partial Q / \partial R$.

If $\mu \neq 0$ and the arbitrary functions are taken constants, the case $\gamma = 1$, one obtains the dust filled magnetofluid model.

The kinematic quantities are

$$\sigma^2 = \frac{1}{2}\sigma_{\alpha\beta}\sigma^{\alpha\beta} = \frac{1}{3}\left(\frac{R\dot{Q} - Q\dot{R}}{RQ}\right)^2 \tag{39}$$

and

$$\theta = \frac{3\dot{R}}{R} - \sqrt{3}\sigma \tag{40}$$

Hence, for $Q = \delta R$ where δ is arbitrary function of *x* the model gives FLRW ones.

CONCLUDING REMARKS

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